The universal R-matrix and the Yang-Baxter equation with parameters

1992 J. Phys. A: Math. Gen. 25 L1023

(http://iopscience.iop.org/0305-4470/25/17/001)

J. Phys. A: Math. Gen. 25 (1992) L1023–L1027. Printed in the UK

LETTER TO THE EDITOR

The universal R-matrix and the Yang–Baxter equation with parameters

Č Burdík† and P Hellinger‡

† Nuclear Centre, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czechoslovakia

‡ Department of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czechoslovakia

Received 11 May 1992

Abstract. We formulate, and demonstrate with some examples, a method for obtaining solutions of the Yang–Baxter equation with parameters.

The Yang–Baxter equation (YBE)

$$R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2) \tag{1}$$

emerges in many branches of theoretical physics (Akutsu and Deguchi 1991).

We will show that it is possible to obtain the solution of the YBE from the irreducible representation (irrep) of the quasi-triangular Hopf algebra, or precisely of its universal R-matrix. The Hopf algebra A is a quasi-triangular Hopf algebra (see Drinfeld 1986), if there exists an invertable element R, $R \in A \otimes A$, called the universal R-matrix, and the comultiplications Δ, Δ' are related by conjugation

$$\Delta'(a) \equiv \sigma\Delta(a) = R\Delta(a)R^{-1} \qquad \sigma(a \otimes b) = b \otimes a \tag{2}$$

for any $a \in A$, and the following conditions are satisfied

$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12} \qquad (\Delta \otimes \mathrm{id})(R) = R_{13}R_{23} \qquad (S \otimes \mathrm{id})(R) = R^{-1}. \tag{3}$$

These above relations (2), (3) imply YBE for the universal R-matrix on the algebra level

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{4}$$

Each central element of the algebra is represented in the irreps by the identity operator (the Schurr lemma) multiplied by a certain number; some of them are determined by the dimension of the irrep (the only case of the deformed enveloping algebra of any simple Lie algebra) and so only the other central elements have arbitrary values, which number the irreps of a certain dimension. Let us denote the set of the values z and the n-dimensional irrep, corresponding to z, as π^n_z. The representation $\pi^n_{z_1} \otimes \pi^n_{z_2}$ of the universal R-matrix R gives us the solution of YBE

$$R^n(z_1, z_2) := (\pi^n_{z_1} \otimes \pi^n_{z_2})R \tag{5}$$

as a straightforward consequence of the properties of the universal R-matrix; applying the representation $\pi^n_{z_1} \otimes \pi^n_{z_2} \otimes \pi^n_{z_3}$ to (4) we get (1).

0305-4470/92/171023+05$04.50

For example, we will investigate the case of $U_{q,s}gl(2)$ (see Schirrmacher *et al* 1991), with the generators $J_0, J_\pm, Z$, with the commutation relations

$$
\begin{aligned}
&[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = \frac{\sinh(2\eta J_0)}{\sinh \eta} \\
&[Z, J_\pm] = 0 \qquad [Z, J_0] = 0 \qquad q = e^\eta
\end{aligned}
\tag{6}
$$

with the comultiplication

$$
\begin{aligned}
&\Delta(J_+) = q^{-J_0}(qs)^Z \otimes J_+ + J_+ \otimes q^{J_0}\left(\frac{s}{q}\right)^Z \\
&\Delta(J_-) = q^{-J_0}(qs)^{-Z} \otimes J_- + J_- \otimes q^{J_0}\left(\frac{s}{q}\right)^{-Z} \\
&\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0 \qquad \Delta(Z) = Z \otimes 1 + 1 \otimes Z
\end{aligned}
\tag{7}
$$

with the antipod

$$
S(Z) = -Z \qquad S(J_0) = -J_0 \qquad S(J_\pm) = -q^{\pm 1} s^{\mp 2Z} J_\pm \tag{8}
$$

and co-unit

$$
\varepsilon(Z) = \varepsilon(J_0) = \varepsilon(J_\pm) = 0. \tag{9}
$$

This algebra has the universal R-matrix $R_{q,s}$ (see Burdík and Hellinger 1992)

$$
R_{q,s} = q^{2(J_0 \otimes J_0 + J_0 \otimes Z - Z \otimes J_0)} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n, q^{-2}]!} (q^{J_0}(qs)^{-Z} J_+)^n \otimes \left(q^{-J_0}\left(\frac{s}{q}\right)^Z J_-\right)^n. \tag{10}
$$

Here and henceforward $[n, q] := (1-q^n)/(1-q)$ and $[n, q]! := [n, q][n-1, q] \ldots 1$.

We will only deal with the two-dimensional representation π_z, as z is an arbitrary complex number

$$
\begin{aligned}
&\pi_z(Z) := \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \qquad \pi_z(J_0) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&\pi_z(J_+) := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \pi_z(J_-) := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{aligned}
\tag{11}
$$

Following the general concept (5), we have this solution of the YBE

$$
R_{q,s}(z_1, z_2) = \begin{pmatrix} qq^{-z_1+z_2} & 0 & 0 & 0 \\ 0 & q^{-z_1-z_2} & 0 & 0 \\ 0 & (q-q^{-1})s^{-z_1+z_2} & q^{z_1+z_2} & 0 \\ 0 & 0 & 0 & qq^{z_1-z_2} \end{pmatrix} \tag{12}
$$

which is simply related to the solution R_5 in Hlavatý (1987).

The second example is more subtle. Let us consider $U_{q,s}gl(1|1)$ (see Bednář *et al* 1992) with the generators $Z, H, \psi^\pm$, with the relations

$$
\begin{aligned}
&[H, \psi^\pm] = \pm 2\psi^\pm \qquad [Z, H] = [Z, \psi^\pm] = 0 \\
&\{\psi^+, \psi^-\} = \frac{q^2 Z - 1}{q^2 - 1} \qquad (\psi^\pm)^2 = 0
\end{aligned}
\tag{13}
$$

with the comultiplication

$$\begin{aligned}
\Delta(\psi_\pm) &= s^{\pm Z/2} \otimes \psi_\pm + \psi_\pm \otimes s^{\mp Z/2} q^Z \\
\Delta(H) &= H \otimes 1 + 1 \otimes H \\
\Delta(Z) &= Z \otimes 1 + 1 \otimes Z
\end{aligned} \tag{14}$$

with the antipod

$$S(Z) = -Z \qquad S(H) = -H \qquad S(\psi^\pm) = -q^{-Z}\psi^\pm \tag{15}$$

and the co-unit

$$\varepsilon(Z) = \varepsilon(H) = \varepsilon(\psi^\pm) = 0. \tag{16}$$

By the quantum double construction (see Drinfeld 1986), more precisely by its graded version, it is possible to find the universal R-matrix in the form

$$R_{q,s} = q^{(H\otimes Z + Z\otimes H)/2} s^{(H\otimes Z - Z\otimes H)/2} (1 \otimes 1 + (1-q^2) s^{-Z/2} \psi^+ \otimes s^{-Z/2} q^{-Z} \psi^-). \tag{17}$$

For the (graded) representation π_z, z being an arbitrary complex number,

$$\begin{aligned}
\pi_z(Z) &:= \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} & \pi_z(H) &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\pi_z(\psi^+) &:= \begin{pmatrix} 0 & (q^{2z}-1)/(q^2-1) \\ 0 & 0 \end{pmatrix} & \pi_z(\psi^-) &:= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \tag{18}$$

We have the solution of the graded YBE, which is simply related to the solution of the YBE of this form:

$$R_{q,s}(z_1, z_2) = \begin{pmatrix}
q^{(z_1+z_2)/2} s^{(-z_1+z_2)/2} & 0 & 0 & 0 \\
0 & q^{(z_1-z_2)/2} s^{(-z_1-z_2)/2} & 0 & 0 \\
0 & q^{(-z_1-z_2)/2}(q^{2z_1}-1) & q^{(-z_1+z_2)/2} s^{(z_1+z_2)/2} & 0 \\
0 & 0 & 0 & -q^{(-z_1-z_2)/2} s^{(z_1-z_2)/2}
\end{pmatrix}. \tag{19}$$

By this construction we may obtain a class of solutions of YBE. However, the main problem is to find a suitable quasi-triangular Hopf algebra because most of the known ones today are the deformed enveloping algebras of the simple Lie algebras. So we were inspired by the case of $U_{q,s}\mathrm{gl}(2)$, which may be thought of as a 'splice' product of $U_q\mathrm{sl}(2)$ and $\mathrm{Uu}(1)$, and we constructed the deformation of the $\mathrm{Usl}(3)\oplus\mathrm{u}(1)^2$ as $U_{q,s_1,s_2}\mathrm{sl}(3)\oplus\mathrm{u}(1)^2$ ($q = \mathrm{e}^h$), with generators $H_1, H_2, X_1^\pm, X_2^\pm, X_3^\pm, Z_1, Z_2$. The algebra structure is as follows:

$$\begin{aligned}
&[H_i, X_i^\pm] = \pm 2X_i^\pm \qquad [H_1, X_2^\pm] = \mp X_2^\pm \qquad [H_2, X_1^\pm] = \mp X_1^\pm \\
&[X_i^+, X_j^-] = \delta_{ij} \frac{\sinh(hH_i/2)}{\sinh(h/2)} \\
&X_3^\pm = q^{1/2} X_1^\pm X_2^\pm - q^{-1/2} X_2^\pm X_1^\pm \\
&0 = q^{-1/2} X_1^\pm X_3^\pm - q^{1/2} X_3^\pm X_1^\pm \\
&0 = q^{1/2} X_2^\pm X_3^\pm - q^{-1/2} X_3^\pm X_2^\pm \\
&[Z_j, H_i] = [Z_j, X_a^\pm] = 0 \qquad i, j = 1, 2;\ a = 1, 2, 3.
\end{aligned} \tag{20}$$

The coalgebra structure is given by:

$$\begin{aligned}
&\Delta H_i = H_i \otimes 1 + 1 \otimes H_i \\
&\Delta Z_i = Z_i \otimes 1 + 1 \otimes Z_i \\
&\Delta X_i^{\pm} = X_i^{\pm} \otimes q^{H_i/2} q^{\pm Z_i/2} s_i^{\pm Z_i/2} + q^{-H_i/2} q^{\mp Z_i/2} s_i^{\pm Z_i/2} \otimes X_i^{\pm} \\
&i = 1, 2.
\end{aligned} \tag{21}$$

The skew antipod has this form:

$$\begin{aligned}
&S_0(H_i) = -H_i \qquad S_0(Z_i) = -Z_i \\
&S_0(X_i^{\pm}) = -q^{\mp 1} s_i^{\mp Z_i} X_i^{\pm} \qquad i = 1, 2.
\end{aligned} \tag{22}$$

The co-unit is given by:

$$\begin{aligned}
&\varepsilon(Z_i) = \varepsilon(H_i) = \varepsilon(X_a^{\pm}) = 0 \\
&i = 1, 2 \qquad a = 1, 2, 3.
\end{aligned} \tag{23}$$

Following the quantum double construction (see Burroughs 1990) we may find the universal R-matrix in the form

$$R_{q,s_1,s_2} = q^{\Sigma_{i,j} a^{-1}_{i,j}(Z_i \otimes H_j - H_i \otimes Z_j + H_i \otimes H_j)} E_{q^{-2}}(\lambda e_1 \otimes f_1) E_{q^{-2}}(-\lambda e_3 \otimes f_3) E_{q^{-2}}(\lambda e_2 \otimes f_2) \tag{24}$$

where a is the Cartan matrix of sl(3), a^{-1} its inverse

$$a = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad a^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{25}$$

and

$$\begin{aligned}
&e_i = q^{H_i/2} q^{Z_i/2} s_i^{-Z_i/2} X_i^{+} \qquad f_i = q^{-H_i/2} q^{Z_i/2} s_i^{Z_i/2} X_i^{-} \qquad e_3 = e_1 e_2 - q^{-1} e_2 e_1 \\
&f_3 = f_1 f_2 - q^{-1} f_2 f_1 \qquad i = 1, 2; \lambda = 1 - q^{-2} \qquad E_x(A) = \sum_{n=0}^{\infty} \frac{1}{[n, x]!} A^n.
\end{aligned} \tag{26}$$

So now we will take the representation as $\pi_{(u_1,u_2)}$, u_1, u_2 being arbitrary complex numbers,

$$\begin{aligned}
&\pi_{(u_1,u_2)}(Z_i) := \begin{pmatrix} u_i & 0 & 0 \\ 0 & u_i & 0 \\ 0 & 0 & u_i \end{pmatrix} \qquad \pi_{(u_1,u_2)}(H_1) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\pi_{(u_1,u_2)}(H_2) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \pi_{(u_1,u_2)}(X_1^{+}) := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\pi_{(u_1,u_2)}(X_1^{-}) := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \pi_{(u_1,u_2)}(X_2^{+}) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&\pi_{(u_1,u_2)}(X_2^{-}) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{27}$$

For this representation we have the solution with four parameters

$$R_{q,s_1,s_2}((u_1, u_2), (v_1, v_2)) = \begin{pmatrix} A & 0 & 0 \\ B & C & 0 \\ D & E & F \end{pmatrix} \tag{28}$$

where $A, \ldots, F, 0$ are 3×3 matrices, 0 is the zero matrix and

$$\begin{aligned}
A &= \begin{pmatrix} q q^{\frac{1}{3}(2u_1+u_2-2v_1-v_2)} & 0 & 0 \\ 0 & q^{\frac{1}{3}(2u_1+u_2+v_1-v_2)} & 0 \\ 0 & 0 & q^{\frac{1}{3}(2u_1+u_2+v_1+2v_2)} \end{pmatrix} \\
B &= \begin{pmatrix} 0 & (q-1/q)s_1^{\frac{1}{2}(-u_1+v_1)} & q^{\frac{1}{6}(u_1+2u_2-v_1-2v_2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
C &= \begin{pmatrix} q^{\frac{1}{3}(-u_1+u_2-2v_1-v_2)} & 0 & 0 \\ 0 & q q^{\frac{1}{3}(-u_1+u_2+v_1-v_2)} & 0 \\ 0 & 0 & q^{\frac{1}{3}(-u_1+u_2+v_1+2v_2)} \end{pmatrix} \\
D &= \begin{pmatrix} 0 & 0 & (q-1/q)s_1^{\frac{1}{2}(-u_1+v_1)} s_2^{\frac{1}{2}(-u_2+v_2)} q^{\frac{1}{6}(u_1-u_2-v_1+v_2)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
E &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (q-1/q)s_2^{\frac{1}{2}(-u_2+v_2)} q^{\frac{1}{6}(-2u_1-u_2+2v_1+v_2)} \\ 0 & 0 & 0 \end{pmatrix} \\
F &= \begin{pmatrix} q^{-\frac{1}{3}(u_1+2u_2+2v_1+v_2)} & 0 & 0 \\ 0 & q^{\frac{1}{3}(-u_1-2u_2+v_1-v_2)} & 0 \\ 0 & 0 & q q^{\frac{1}{3}(-u_1-2u_2+v_1+2v_2)} \end{pmatrix}.
\end{aligned} \tag{29}$$

In this letter we have constructed some solutions of the YBE and we propose this method for the construction of new ones, intimately connected with some quasi-triangular Hopf algebras. Note that none of the solutions (12), (19), (28) is a function of the difference, so it is not possible to find these solutions by the Baxterization process (see Cheng *et al* 1991).

References

Akutsu Y and Deguchi T 1991 *Phys. Rev. Lett.* **67** 777

Bednář M, Burdík Č, Couture M and Hlavatý L 1992 *J. Phys. A: Math. Gen.* **25** L341

Burdík Č and Hellinger P 1992 *J. Phys. A: Math. Gen.* **25** L629

Burroughs N L 1990 *Commun. Math. Phys.* **127** 109

Cheng Y, Ge M-L and Xue Y 1991 *Commun. Math. Phys.* **136** 195

Drinfeld V G 1986 *Proc. IMS* (Berkeley Academic Press) 1 798

Hlavatý L 1987 *J. Phys. A: Math. Gen.* **19** L1661

Schirrmacher A, Wess J and Zumino B 1991 *Z. Phys.* C **49** 317